① Recall that $E(A,B) \leq \min(|A||B|^2, |A|^2|B|)$.

If $|B| > K^2|A|$, then

$$e(A,B) = \frac{E(A,B)}{(|A||B|)^{3/2}} \leq \left(\frac{|A|}{|B|}\right)^{1/2} < \frac{1}{K}.$$

Similarly, if $|B| < \frac{1}{K^2}|A|$,

$$e(A,B) \leq \left(\frac{|B|}{|A|}\right)^{1/2} < \frac{1}{K}.$$

② $$\mu(1_A * 1_B) = \sum_{g\in G} \overbrace{1_A * 1_B(g)}^{r_{A,B}(g)}$$

$$= \sum_{g\in G} \sum_{g_1 g_2 = g} 1_A(g_1) 1_B(g_2)$$

$$= \sum_{g_2\in G} 1_A(g_1) \sum_{g_2 \in G} 1_B(g_2) = \mu(A)\mu(B)$$

$$E(A,B) = \sum_{g\in G} r_{A,B}(g)^2 = \sum_{g} 1_A * 1_B(g)^2$$

$$= \mu((1_A * 1_B)^2).$$

since $1_A * 1_B(g) \geq 0$, $\forall g \in G$.

③ Using Cauchy-Schwarz, we have

$$E(A,B) = \sum_{g\in A\cdot B} r_{A,B}(g)^2 \geq \frac{\left(\sum_{g\in A\cdot B} r_{A,B}(g)\right)^2}{\sum_{g\in A\cdot B} 1}$$

$$= \frac{\left(\sum_{g\in G} r_{A,B}(g)\right)^2}{\mu(A\cdot B)} = \frac{\mu(A)^2\mu(B)^2}{\mu(A\cdot B)}$$

(we used $r_{A,B}(g)=0$ if $g\notin A\cdot B$).

A simple application shows that

$$e(A,B) = \frac{E(A,B)}{(\mu(A)\mu(B))^{3/2}} \geq \frac{\mu(A)\mu(B)^{1/2}}{\mu(A\cdot B)} \geq \frac{1}{K}$$

if $\mu(A\cdot B) \leq K\mu(A)^{1/2}\mu(B)^{1/2}$.

④ Let $A,B\subset G$ with $e(A,B)\geq \frac{1}{K}$.

From BSG set-theoretic version, $\exists A'\subset A, B'\subset B$, an absolute constant C such that

$$|A'| \geq \frac{1}{CK}|A|,\quad |B'| \geq \frac{1}{CK}|B|,$$

$$\mu(A'\cdot B') \leq CK^8\mu(A)^{1/2}\mu(B)^{1/2}.$$

Now from subgroup recognition criterion,

$\exists\ C_1, d,\ K' \leq C_1 K^d$, H a K'-approx subgp, $a, b \in G$ s.t.

$|A'| \leq K' |A' \cap aH|$,

$|B'| \leq K' |B' \cap Hb|$, $\quad |H| \leq K' |A|$.

Hence $|A| \leq CK|A'| \leq CKK'|A' \cap aH|$

$\leq CKK'|A \cap aH|$.

Similarly for B. (using $A' \cap aH \subseteq A \cap aH$)

5. (i) $\Rightarrow$ (iii) is BSG set-theoretic

Note that using exercise 1,

$$(i) \Rightarrow \mu(A) \approx_{K^{O(1)}} \mu(B).$$

(i) $\Rightarrow$ (iv) is BSG approx-gp version.

(iii) $\Rightarrow$ (iv) Note that the trivial bound $\mu(A'.B') \geq \max(\mu(A'), \mu(B'))$ implies

$$\mu(A) \approx_{K^{O(1)}} \mu(B).$$

By approx subgp criterion, we find a $O(K^{O(1)})$ approx gp H, $\mu(H) \ll K^{O(1)} \mu(A)$, set X with $\mu(X) \ll K^{O(1)}$ s.t.

$$A' \subseteq X.H, \quad B' \subseteq H.X.$$

Now use pigeonhole to find $x, y \in X$ s.t.

$$\mu(A' \cap (xH)) \gg K^{O(1)} \mu(A)$$

$$\mu(B' \cap (Hy)) \gg K^{O(1)} \mu(B).$$

(iv) $\Rightarrow$ (ii) Take $E = (A \cap xH) \times (B \cap Hy)$.

$$\mu(\{a.b : (a,b) \in E\}) \le \mu(x.H.Hy)$$

$$= \mu(H^2) \ll K^{O(1)} \mu(H) \ll K^{O(1)} \mu(A)^{1/2} \mu(B)^{1/2}.$$

(ii) $\Rightarrow$ (i)

Let $C = \{ab : (a,b) \in E\}$. So $\mu(C) \ll K^{O(1)} (\mu(A)\mu(B))^{1/2}$

Note that $\displaystyle\sum_{g \in C} 1_A * 1_B(g) = \sum_{g \in C} \sum_{g_1 g_2 = g} 1_A(g_1) 1_B(g_2)$

$$\ge \sum_{(g_1, g_2) \in E} 1. \gg K^{O(1)} \mu(A) \mu(B).$$

Now $\displaystyle E(A,B) = \sum_{g \in G} (1_A * 1_B(g))^2 \ge \sum_{g \in C} (1_A * 1_B(g))^2$

$$\ge \Big(\sum_{g \in C} 1_A * 1_B(g)\Big)^2 \Big/ \mu(C)$$

$$\gg K^{O(1)} \mu(A)^{3/2} \mu(B)^{3/2}$$

✓.